ON FLOWS IN THE REGION OF THE TRANSITION SURFACE

(O TECHENIIAKH V OKRESTNOSTI POVERKHOSTI PEREKHODA V SOPLAKH LAVALIA)

PMM Vol.22, No.4, 1958, pp.433-443

O. S. RYZHOV
(Moscow)

(Received 5 January 1958)

The peculiarity of transition through the velocity of sound in a plane nozzle, i.e., in the case when the tangent to the sonic line coincides with the direction of characteristics which pass through the axis of the channel, was pointed out by Khristianovich [1]. Later Frankl, on the basis of a hodograph transformation, investigated in detail the character of a plane stream in the vicinity of the sonic line [2]. Applying a direct method Falkovich obtained the main term of the solution in the form of a third degree polynomial which considerably simplified all the calculations of the transition region of a nozzle [3]. In the present work on the basis of this solution, some properties of flow with axial symmetry are investigated. The investigation is based on the method of Falkovich, since, in this case, it is not necessary to make use of the hodograph transformation.

1. Analytical nozzles. The equations of transonic flow of a gas with axial symmetry in a cylindrical coordinate system have the form:

$$-(\varkappa+1)UU_x + a_* V_r + a_* V/r = 0, \qquad U_r = V_x \tag{1.1}$$

where U and V are the additions to the velocity along the axes x and r which is equal to the critical velocity a and which is directed along x-axis; κ is Poisson's adiabatic index; the subscripts denote partial differentiation.

It is well known that the flow near the throat of a nozzle is not necessarily free from discontinuities in density [1,2]. Therefore in the computation of a nozzle initial conditions are given in terms of an analytic velocity distribution along the nozzle axis rather than in terms of prescribed wall shape. From the equations of motion the velocity potential corresponding to these functions is determined. Two stream lines in this flow field which are symmetrical with respect to the x-axis, are then taken as the walls of the nozzle.

It should be noted now that in going over from exact equations of gas dynamics to the simplified equations (1.1) we have to deal in essence with linear theory, though the equations (1.1) are not linear. But it follows from this that the quantity:

$$U_x = C_0 \quad \text{при} \;\; x = 0, \;\; r = 0 \qquad (V = 0 \quad \text{при} \;\; r = 0) \tag{1.2}$$

is a unique numerical parameter which describes the flow along the nozzle axis.

In the paper [4] the system of equations (1.1) is shown to be invariant under the continuous group of transformations:

$$U_*(x, r) = \alpha^{2(n-1)} U(\alpha x, \alpha^n r), \qquad V_*(x, r) = \alpha^{3(n-1)} V(\alpha x, \alpha^n r) \tag{1.3}$$

(where a and n are arbitrary constants). If in the formulas (1.3) we take $n = 1/2$, then the initial data (1.2) will also be invariant with respect to the indicated group of similarity transformations. Hence we conclude that the values $r^{-2}U$ and $r^{-3}V$ can be functions only of one variable, namely $\xi = xr^{-2}$. Thus the flow in an analytical nozzle in the neighborhood of its center will be self similar. To determine it we assume:

$$u = (\varkappa + 1) U / a_*, \qquad v = (\varkappa + 1) V / a_* \tag{1.4}$$

Then equations (1.1) take the form:

$$-uu_x + v_r + v / r = 0, \qquad u_r = v_x \tag{1.5}$$

Conforming to the outline above we look for a solution of the system of equations (1.5) in the form:

$$u = r^2 f(\xi), \qquad v = r^3 g(\xi), \qquad \xi = x / r^2 \tag{1.6}$$

The functions f and g satisfy the system of ordinary differential equations

$$ff' - 4g + 2\xi g' = 0, \qquad 2\xi f' + g' - 2f = 0 \tag{1.7}$$

Eliminating the function g from the system (1.7) we obtain, for the determination of function f, a second order differential equation:

$$(f - 4\xi^2) f'' + f'^2 + 4\xi f' - 4f = 0 \tag{1.8}$$

Equation (1.8) has a simple particular solution, which we shall call the basic solution,

$$f = \tfrac{1}{4} A^2 + A\xi \tag{1.9}$$

where A is an arbitrary constant, equal in magnitude to the derivative u_x at the center of a nozzle. Using equations (1.6) and (1.0) we find

$$u = Ax + \frac{1}{4} A^2 r^2, \qquad v = \frac{1}{2} A^2 xr + \frac{1}{16} A^3 r^3 \tag{1.10}$$

Formulas (1.10) describe the flow in the neighborhood of the transition surface in nozzles of revolution. Assuming the velocity to increase along the x-axis, we have $A > 0$.

2. Investigation of the flow in the neighborhood of the nozzle center. One of the properties of flow in plane Laval nozzles, as determined by Frankl, is the many-valued character of the transformation of the physical plane in the hodograph plane in the neighborhood of the sonic line. It is found that in a complete circuit around the origin of coordinates the region between two characteristics in the hodograph plane is traversed three times [2]. We shall show in this section that in the case of axial symmetry also there is not singlevalued correspondence between the xr and uv planes. Indeed, let us look at the Jacobian $j = \partial(u,v)/\partial(x,r)$ of the transformation (1.10). Equating this Jacobian to zero determines the branch line (L-curve) in the xr plane, along which the one-to-one correspondence between the physical plane and the hodograph plane is violated. Since $j = 1/2\, A^3(x - 1/8\, Ar^2)$, the equation of the L-curve will be

$$x = \frac{1}{8} Ar^2 \tag{2.1}$$

Hence it follows that, as in the case of the plane nozzle, the L-curve is concave toward the oncoming stream. The transformation of the branch line in the hodograph plane is S-curved, the equation of which is written in the form

$$u = \frac{3}{2} v^{2/3} \tag{2.2}$$

To clarify the position of the branch line in the physical plane and in the hodograph plane, let us find, in addition, the equations of the characteristics which pass through the nozzle center, and also the equations of the lines $u = 0$ and $v = 0$. The characteristics are determined by the differential equation

$$(dx/dr)^2 = u = Ax + \frac{1}{4} A^2 r^2 \tag{2.3}$$

These give the equations of the characteristics which pass through the nozzle center and which are tangent to the sonic line; in future such characteristics will be called "singular" characteristics. They are

$$\begin{aligned} x &= \frac{1}{8} A (1 + \sqrt{5}) r^2 \qquad (c^\circ_+\text{- characteristic}) \\ x &= \frac{1}{8} A (1 - \sqrt{5}) r^2 \qquad (c^\circ_-\text{- characteristic}) \end{aligned} \tag{2.4}$$

Substituting formulas (2.4) into equations (1.10), the characteristics in the hodograph plane (Γ - characteristics) are obtained.

$$u = 4^{1/3} v^{2/3} \tag{2.5}$$

From formulas (2.5) it follows that both c^0 - characteristics in the physical plane transform into different branches of the same semicubical parabola in the hodograph plane. Let us write now the equation of the sonic line; in the hodograph plane the sonic line is the axis $u = 0$; in the physical plane its equation will be

$$x = -\tfrac{1}{4} Ar^2 \tag{2.6}$$

The equation of the line on which the radial velocity component vanishes in the hodograph plane has the form $v = 0$, while in the physical plane:

$$x = -\tfrac{1}{8} Ar^2 \tag{2.7}$$

It is seen from formulas (2.6) and (2.7) that the sonic line and the line $v = 0$ are concave, as in plane nozzles, with respect to the supersonic flow. It follows from equations (2.1) and (2.7) that in the case of flow with axial symmetry the radius of curvature of the branch line is not equal to the radius of curvature of the sonic line, instead it is equal to the radius of curvature of the line along which the velocity of the stream is parallel to the nozzle axis.

The neighborhood of the nozzle center is shown in Fig. 1 where the relative positions of the sonic line of the characteristics, of the line of zero radial velocity and of the branch line are shown. As the values of r can only be greater than, or equal to zero, only the upper half of the physical plane needs to be considered. A transformation of the neighborhood of the origin of coordinates in the uv plane, which has the form of a folded surface, is shown in Fig. 2. Corresponding regions in Fig. 1 and 2 are denoted by the same numbers. Regions IV, V and VI of the xr plane map into the same single region of the uv plane.

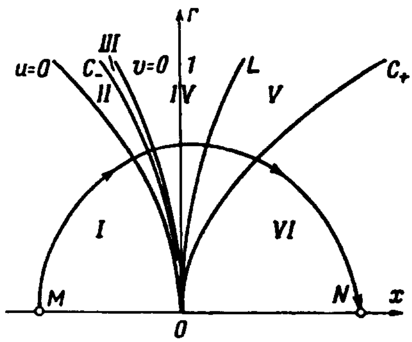

Fig. 1.

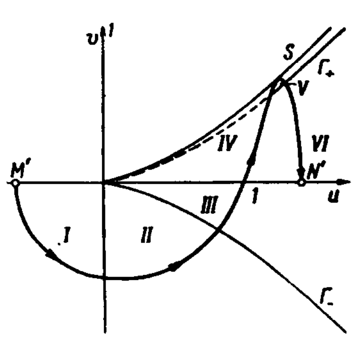

Fig. 2.

Note: It should be noted that the branch line in the case considered does not coincide with the characteristic which passes through the nozzle center, as was the case in two dimensional flows. This is due to the fact that the equations of motion in the case of axial symmetry are irreducible owing to the presence of an additional term in the equation of continuity.

As formulas (2.4), (2.6) and (2.7) show, the corresponding curves in the case of axial symmetry are situated closer to the vertical line than in the plane case if in both flows the derivative u_x at the nozzle center is the same. The flow in a nozzle of revolution is therefore more uniform than in a plane nozzle. From this point of view the construction of nozzles of revolution is to be favored over that of plane nozzles with the same rate of acceleration from subsonic to supersonic flow.

3. Nozzles with surfaces of weak discontinuities. 1. Consider now the case when weak discontinuities, i.e. discontinuities of first derivatives of velocity components are formed along the Mach lines, originating from the center of the nozzle. For this purpose we shall assume that A in formulas (1.10) in regions I and II is equal to some value A_1, while in region IV it is equal to some other value A_2, where $A_1 \neq A_2$. Assuming that limiting values of the derivative u_x at the nozzle center when approached from both left and right are positive, we find that $A_1 > 0$ and $A_2 > 0$. Then in regions I and II we shall have:

$$u = A_1 x + \tfrac{1}{4} A_1 r^2, \qquad v = \tfrac{1}{2} A_1^2 x r + \tfrac{1}{16} A_1^3 r^3 \tag{3.1}$$

and correspondingly, in region VI

$$u = A_2 x + \tfrac{1}{4} A_2 r^2, \qquad v = \tfrac{1}{2} A_2^2 x r + \tfrac{1}{16} A_2^3 r^3 \tag{3.2}$$

The equations of the c^0 - characteristics, using equations (2.3), (3.1) and (3.2), can be written in the form

$$\begin{aligned} x &= \tfrac{1}{8} A_1 (1 - \sqrt{5})\, r^2 \qquad c^\circ_- \text{- characteristic} \\ x &= \tfrac{1}{8} A_2 (1 + \sqrt{5})\, r^2 \qquad c^\circ_+ \text{- characteristic} \end{aligned} \tag{3.3}$$

We now note that the equations obtained coincide with the lines ξ h ξ = const (see Section 1). Therefore the solution of the equations of motion (1.5) in regions III, IV and V can be expected, as before, to be of the form (1.6), reducing the problem in this manner to the integration of an ordinary differential equation (1.8). For the c^0 - characteristics the following equations apply:

$$\begin{aligned} u &= \tfrac{1}{8} A_1^2 (3 - \sqrt{5})\, r^2 \qquad \text{on } c^\circ_- \text{- characteristic} \\ u &= \tfrac{1}{8} A_2^2 (3 + \sqrt{5})\, r^2 \qquad \text{on } c^\circ_+ \text{- characteristic} \end{aligned} \tag{3.4}$$

Using formulas (1.6), (3.3) and (3.4) two limiting conditions for the integration of equation (1.8) are obtained:

$$\begin{aligned} f = f_1 &= \tfrac{1}{8} A_1^2 (3 - \sqrt{5}) \qquad \text{for } \xi = \xi_1 = \tfrac{1}{8} A_1 (1 - \sqrt{5}) \\ f = f_2 &= \tfrac{1}{8} A_2^2 (3 + \sqrt{5}) \qquad \text{for } \xi = \xi_2 = \tfrac{1}{8} A_2 (1 + \sqrt{5}) \end{aligned} \tag{3.5}$$

Since $A_1 > 0$ and $A_2 > 0$, it follows that from equations (3.5)

$$\xi_1 < 0, \qquad \xi_2 > 0 \tag{3.6}$$

2. The method of investigation of the formation of weak discontinuities along a Mach line can be simplified further if we make use of the invariance of equation (1.8) with respect to the group of similarity transformations $\Phi(\xi) = a^{-2}f(a\xi)$, where a is any constant not equal to zero. Assuming therefore

$$f = \xi^2 F(\eta), \qquad \frac{dF}{d\eta} = \Psi, \qquad \eta = \ln|\xi| \tag{3.7}$$

equation (1.8) is written in the form

$$\frac{d\Psi}{dF} = \frac{\Psi^2 + 7\Psi F + 6F^2 - 8\Psi - 4F}{\Psi(4-F)} \tag{3.8}$$

The basic problem is now reduced to the investigation of equation (3.8). The general picture of the field of its integral curves is illustrated in Fig. 3. This graph clarifies the character of singular points of the curves Ψ_1^* and Ψ_2^*, at which the value of the derivative $d\Psi/dF$ is equal to zero, and also the character of the lines $F = 4$ and $\Psi = 0$, where the derivative $d\Psi/dF$ becomes infinite.

For our purpose we need to know the location of the points

$$A(0, 0), \qquad C[4, -2\sqrt{5}(\sqrt{5}-1)], \qquad D[4, -2\sqrt{5}(\sqrt{5}+1)]$$

and of the particular point E which recedes to infinity and which is reached by going down along the line $\Psi = -2F$. It can be shown that the point A corresponds to the x-axis; the point C corresponds to the c^0 - characteristic, while the point D corresponds to the c^0 - characteristic determined by equation (3.3); the point E corresponds to the r-axis. From equations (1.6) and (3.6) it follows that the ordinate corresponds to the sonic line, one half of the plane to the right of this axis corresponds to the region of supersonic velocities and the left half of the plane corresponds to subsonic velocities. When moving along some integral curve in the $F\Psi$ plane, the corresponding lines ξ = const will describe a certain region in physical space. Hence it is clear that ξ must not have any extreme values, because otherwise we would get a multivalued physical xr plane, in which the flow will be superimposed upon itself. A line on which the value ξ is stationary is a limit line. Using equations (3.7) and (3.8) it is easily seen that from this point of view transition through the line $F = 4$ is impossible. The only exceptions are those integral curves which pass through the particular points C and D.

Using formulas (3.7) we have

$$f' = \xi(2F + \Psi) \tag{3.9}$$

From this equation we can obtain the equation of the integral curve K which is the transformation of the basic solution (1.9) in the $F\psi$ plane; this equation does not depend on the constant A

$$\Psi = -2\left(1 + F \mp \sqrt{1+F}\right) \qquad (3.10)$$

For motion in the physical plane from the subsonic region to the supersonic region the motion along the curve (3.10) will be in the direction indicated by the arrow in Fig. 3.

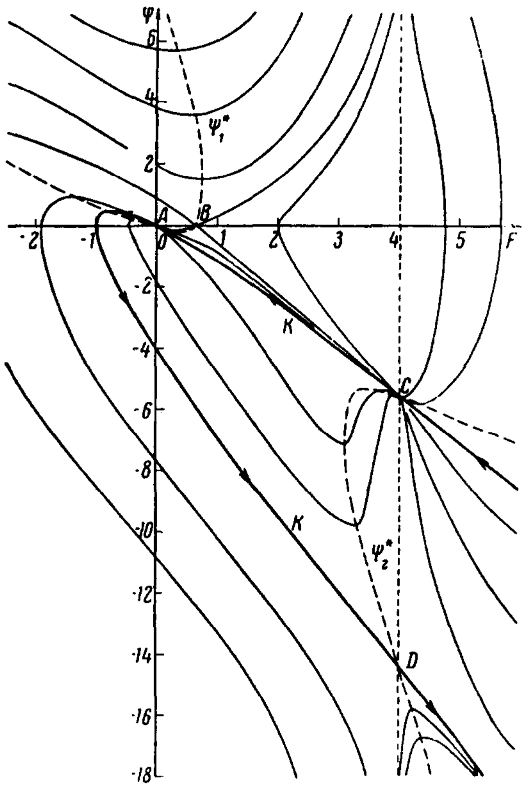

Fig. 3.

Consider now flows with weak discontinuities along characteristics originating at the center of the nozzle. In this case the flow in regions I and II will be, as before, represented by the segment of the K-curve between points A and D, while the flows in region VI correspond to the segment of the K curve between the points C and A, because equation (3.10) does not depend on values A_1 and A_2. On the characteristics the values of f are continuous, but the values of f' have discontinuities. Hence it follows that the values F must also be discontinuous and equal to 4, while the values ψ must undergo discontinuities of the first kind. Therefore when moving along the segment of the K-curve in the direction away from the subsonic region and reaching point D, we have a unique possibility to realize flow with weak discontinuities jumping from point D to point C. From the point C we can move along any integral curve that is bound by the two branches of the K-curve to the point E, and then return along the continuation of this curve to the point C again. In the limiting case the motion will take place along the K-curve which goes in the opposite direction; namely by jumping from the point D to the point C, continuing from this point along the K-curve to the point E, then along its continuation to the point D and again jumping to point C.

From the investigation of the field of the integral curves of equation (3.8) we can derive several interesting properties about flows with weak discontinuities.

Since the flow past the ${c^0}_+$ - characteristic is mapped in the interval of the K-curve between points C and A, it follows that there are no discontinuities in the first order derivatives of the velocity components on the ${c^0}_+$ - characteristic, although they did arise on the ${c^0}_-$ - characteristic. Thus, weak discontinuities (i.e. discontinuities in first derivatives) do not reflect from the nozzle center even in the case when the nozzle is axially symmetrical. This property is derived from the degeneracy of the equations of motion at this point into equations of parabolic type. An exception to this behavior is the limiting case, when the flow between singular characteristics reflects in the K-curve extending in the opposite direction. In this case weak discontinuities form on both the ${c^0}_-$ - characteristic and the ${c^0}_+$ - characteristic. If we move from the point C along the integral curve situated above the dividing curve K, then f' for the ${c^0}_-$ - characteristic becomes infinite.

3. We now clarify the character of the integral curves of equation (1.8) which correspond to the curves in the $F\psi$ plane considered above. First it is to be noted that the magnitude of the discontinuity in the derivative u_x for the ${c^0}_-$ - characteristic may not be an arbitray quantity. Indeed, computing from formula (3.9) the value of f'so we approach the ${c^0}_-$ - characteristic from the left, we have

$$f_1' = -\frac{A_1}{2}(3-\sqrt{5}) \tag{3.11}$$

Hence, denoting the magnitude of the discontinuity in the derivative u_x across the ${c^0}_-$ - characteristic by $[u_x]$, we get

$$[u_x] = -\frac{1}{2}A_1(5-\sqrt{5}) < 0$$

Thus, if we assume a given flow on the left side of the ${c^0}_-$ characteristic, we can obtain a different flow on its right side. However, the value of the jump $[u]$ across the line of contact of the two flows is constant.

Let us consider now an integral curve of equation (1.8) corresponding to some curve in the $F\psi$ plane, originating and terminating at the point C. On the c^0_+ - characteristic, when approached from the right (and from the left), we have

$$f_2' = A_2 \tag{3.12}$$

Equations (3.5), (3.11) and (3.12), which have to be satisfied at the limits of integrating for equation (1.8), may be written in the form

$$\begin{aligned} f_1 &= 4\xi_1^2, \qquad f_1' = 2(\sqrt{5}-1)\xi_1 \quad \text{where } \xi = \xi_1 \\ f_2 &= 4\xi_2^2, \qquad f_2' = 2(\sqrt{5}-1)\xi_2 \quad \text{where } \xi = \xi_2 \end{aligned} \tag{3.13}$$

The reduction of the last four equations is made possible by the fact that the limits of the interval $P(\xi_1, f_1)$ and $Q(\xi_2, f_2)$ are the singular points of equation (1.8), through which pass an infinite number of integral curves with the same slope. Indeed, computing the roots f'^* of the equation $f'^2 + 4\xi f' - 16\xi^2 = 0$, we have

$$f_{1,2}'^* = 2(\pm\sqrt{5}-1)\xi \tag{3.14}$$

This also follows from the fact that the point C is a singular point of equation (3.8). Using inequalities (3.6), it follows from formulas (3.13) that $f_1' < 0$, $f_2' > 0$, i.e. the function f is not monotonic in the interval considered. Correspondingly the magnitude of the velocity u at first decreases but then increases along lines r = const in the region between the singular characteristics.

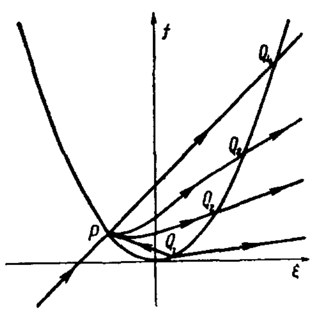

Fig. 4.

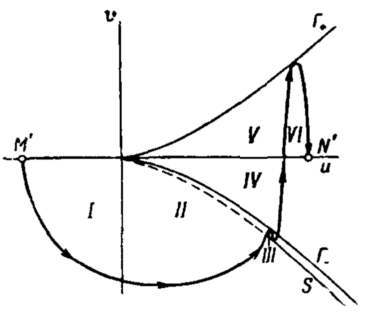

Fig. 5.

We now investigate the limiting case when the flow in the $F\psi$ plane is given by the K-curve extending in the opposite direction. In this case equation (1.8) has a simple solution:

$$f = \tfrac{1}{8}A_1^2(7-3\sqrt{5}) - \tfrac{1}{2}A_1(3-\sqrt{5})\xi \tag{3.15}$$

Quantity A_2 is expressed in terms of constant A_1 by the formula

$$A_2 = \tfrac{1}{2}(7-3\sqrt{5})A_1 \tag{3.16}$$

From formula (3.15) it follows that in this limiting case f' is less than zero everywhere, i.e. the velocity along r = const in the region of flow bounded by the singular characteristics, decreases monotonically.

Since the flows mapped in the $F\psi$ plane by the portions of the K-curve extending in the forward and backward directions are finite, it is easy to establish the inequalities

$$1 \leqslant \frac{A_1}{A_2} \leqslant \frac{2}{7-3\sqrt{5}} \tag{3.17}$$

The picture of the field of integral curves in the ξf is represented in Fig. 4. As follows immediately from equations (3.13) both boundary points $P(\xi_1, f_1)$ and $Q(\xi_{2i}, f_{2i})$ lie on the parabola $f = 4\xi^2$. The line $f = 1/4\, A_1^2 + A_1\xi$, corresponding to continuous flow, and the line $f = 1/8\, A_1^{\,2}(7 - 3\sqrt{5}) - 1/2\, A_1(3 - \sqrt{5})\,\xi$, corresponding to flow with discontinuities in the first derivatives on both singular characteristics, are finite. All the other integral curves of equation (1.8) which have their origin at the point p and bounded derivative at the limits of the interval considered are situated between them.

It is interesting to point out that in the case of flow with weak discontinuities on both singular characteristics the lines u = const are concave toward the oncoming stream in the region between the two characteristics.

The line of zero radial velocity is also concave in the direction of subsonic velocities and is given by the equation

$$x = \frac{3-\sqrt{5}}{16} A_1 r^2$$

The branch line, which is defined by the formula

$$x = -\frac{3-\sqrt{5}}{16} A_1 r^2$$

is, in contrast, concave with respect to the supersonic flow. Besides, it is easy to convince oneself that the Jacobian changes sign also on both singular c^0 - characteristics. Therefore the mapping of the neighborhood of the nozzle center in the uv plane will be considerably more complicated in this case than in the case of the flow in analytical nozzles; the latter was shown in Fig. 5.

The flow patterns in the physical plane for the cases of continuous flow and flow with possible finite weak discontinuities are represented in Figs. 6 and 7 respectively.

It is seen in these Fig. that the shape of the throat of the nozzle (i.e. its narrowest part) in the case of the discontinuities in the derivatives of the characteristics is elongated considerably.

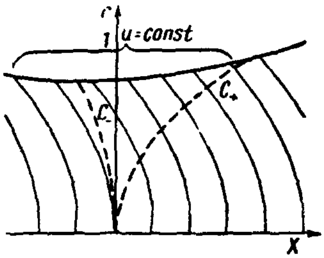

Fig. 6.

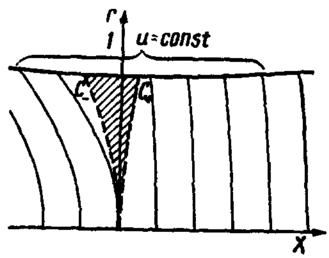

Fig. 7.

4. Plane nozzles. Let us consider briefly some of the properties of flows in plane nozzles. In this case the term v/r must be deleted from the equation of continuity. The solution of the equation of motion can be looked for, as in the preceding investigation, in the form (1.7). The equation analogous to equation (1.8), will take for form [3]:

$$(f - 4\xi^2) f'' + f'^2 + 2\xi f' - 2f = 0 \tag{4.1}$$

Falkovich has derived a general integral of this equation [3].

It seems useful, however, to make use of the method of the "phase" $F\psi$ plane again, mentioned in Section 3, so that some of the properties of the flows investigated may be clarified.

Since equation (4.1) differs from the equation (1.8) only in the coefficients of the last two terms, it is also invariant with respect to the group of transformations defined above. Making use of formulas (3.7), it can be reduced to the form

$$\frac{d\Psi}{dF} = \frac{\Psi^2 + 7\Psi F + 6F^2 - 10\Psi - 6F}{\Psi(4 - F)} \tag{4.2}$$

The general picture of the field of integral curves of (4.2) is represented in Fig. 8, where the notation of the last section is used. The basic solution of equation (4.1) can be represented in the form [3]

$$f = \tfrac{1}{2}A^2 + A\xi \tag{4.3}$$

As before, we shall call the representation of the basic solution in the $F\psi$ plane the K-curve; its equation will be

$$\Psi = -(1 + 2F \mp \sqrt{1 + 2F}) \tag{4.4}$$

Motion along the K-curve in the direction indicated by the arrow (Fig. 8) corresponds to flow in an analytical nozzle. As before, flows with discontinuities in the first derivatives of the velocity components across Mach lines are represented in the $F\psi$ plane by the curves originating and terminating in the point C. In the limiting case the flows with weak discontinuities will map into the K-curve which goes in the opposite direction. If we move from the point C along an integral curve situated above the

K-curve, then for the corresponding flow in the physical plane the function f' becomes infinite.

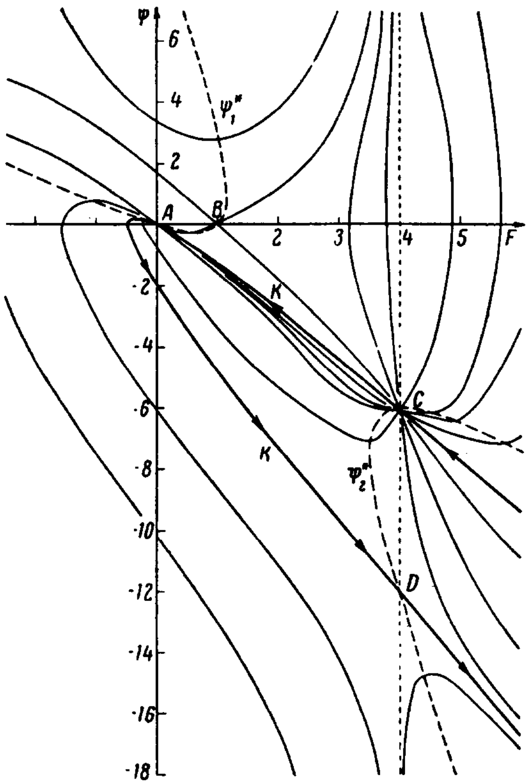

Fig. 8.

Hence it is easy to derive the properties, which were first pointed out by Frankl [2], and which, as it was shown above, occurred in the cases of the stream in nozzles of revolution. Repeating the arguments of Section 3 it can be concluded that, generally speaking, the weak discontinuities along the Mach line, originated along the nozzle center and directed downstream, do not arise. The only exception is the limiting case of flow which maps onto the K-curve extending in the opposite direction; in such a flow the discontinuities in the first derivatives occur on both c^0-characteristics. In this case the solution of the equation (4.1) has the form:

$$f = \tfrac{1}{8} A_1^2 - \tfrac{1}{2} A_1 \xi \tag{4.5}$$

The constant A_2 is expressed then in terms of A_1 by the equation

$$A_2 = \tfrac{1}{4} A_1 \tag{4.6}$$

Hence we have [2]

$$1 \leqslant A_1/A_2 \leqslant 4 \tag{4.7}$$

Also it is easily shown that in the case of the flows represented in the $F\psi$ plane by the curve originating and terminating in the point C,

$$f_1' = 2\xi_1 < 0, \qquad f_2' = 2\xi_2 > 0$$

Hence it follows, as before, that the velocity along the line r = const at first decreases, but then increases only in the region between the Mach lines originating at the center of the nozzle. In the case of flow with discontinuities in first derivatives on both c^0-characteristics the velocity in the region between them decreases monotonically, as is readily seen from equation (4.5). Further, the magnitude of the discontinuity $[u_x]$ on the c^0-characteristic, if the constant A_1 is assumed, is expressed uniquely in terms of it. The behavior of the solutions of equation (4.1) will be qualitatively the same as that of the curves shown in Fig. 4, while patterns in the physical xr plane of flows in analytical nozzles and in nozzles with weak discontinuities on both singular c^0-characteristics will be similar to those shown in Fig. 6 and 7. The character of the transformation from the xr plane to the uv plane in all the cases under consideration will be the same, because in the case of plane gas flows there exists in the hodograph plane a fixed net of characteristics, which does not depend on the solution of equations (1.1). From the character of the integral curves in the $F\psi$ plane in the neighborhood of the point C it is seen that discontinuities of the first derivatives on the c^0-characteristic (which we have called weak discontinuities everywhere) will reflect from the center of a plane nozzle along the c^0_+ - characteristic in the form of the discontinuities in the second derivatives of the fluid velocity components. The weak discontinuities along the c^0_+ - characteristic will reflect from the center of a nozzle of revolution in the form of discontinuities in the third derivatives. In this sense the statement that the weak discontinuities are not reflected from the center of a nozzle, which was made above, is not exact.

In conclusion we note the property, peculiar to plane flow, that the c^0 - characteristics $x = 1/2\, Ar^2$ coincide with the branch line. From equation (2.3) it is possible to deduce that the derivative d^2r/dx^2, which determines the curvature of the c-characteristics, becomes zero on the line L. Therefore every c^0-characteristic belonging to one family has its point of inflection on the above singular c^0-characteristic of another family. This property is due to the fact that in functions $x(u,v)$, $r(u,v)$ are not single-valued in the region bounded by the branches of the semicubic parabola $v^2 = 4/9\, u^3$. For each branch of the functions $x(u,v)$, $r(u,v)$ in this region the sense of concavity of the c - characteristics does not change along the entire length of the branch line, which corresponds to the results of Khristianovich [1].

Each of the singular characteristics $r = \pm\sqrt{2x/A}$ is also a locus of points at which the quantities u and v attain extreme values along the non-singular c-characteristics. Since the mapping of the physical plane on the hodograph plane is not one-to-one, the Γ - characteristics of one of the families extend in both forward and backward directions. This property is derived from the fact that the c^0-characteristic of one family is the last characteristic which connects any non-singular c - characteristic of the other family with the sonic line [5].

The author appreciates many discussions with S.A. Khristianovich which stimulated the present work.

BIBLIOGRAPHY

1. Khristianovich, S.A., O sverkhevukovykh techeniiakh gaza (On supersonic gas flows). *Trud. Tsent. Aero-gidrodin. Inst. Mosk.* No. 543, 1941.

2. Frankl', F.I., K teorii sopel Lavalia (On the theory of Laval nozzles) *Izv. Akad. Nauk USSR Ser. mat.* Vol. 9, No. 5, 1945.

3. Falkovich, S.V., K teorii sopel Lavalia (On the theory of Laval nossles). ***PMM*** Vol. 10, No. 4, 1946.

4. Ryzhov, O.S., Nekotorye vyroshdennye okolozvukovye techeniia (Certain degenerate transonic flows). ***PMM*** Vol. 22, No. 2, 1958.

5. Nikol'skii, A.A. and Taganov, G.I., Drizhenie gaza v mestnoi sverkhzvukoroi zone i nekotorye usloviia razrusheniia potentsial'nogo techeniia (Gas flow in a local supersonic region and conditions for the breakdown of potential flow). ***PMM*** Vol 10, No. 4, 1946.

Translated by J.R.W.